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GROUPS OF PL Λ-HOMOLOGY SPHERES

BY

GERALD A. ANDERSON[1]

ABSTRACT. Let $\Lambda = \mathbf{Z}_K$ be a subring of $\mathbf{Q}$. The group of H_K-cobordism classes of closed PL n-manifolds with the Λ-homology of S^n is computed for $n > 4$ (modulo K-torsion). The simply connected version is also computed.

1. Introduction. Let K be a set of primes and $\Lambda = \mathbf{Z}[1/p\colon p \in K]$ the localization of the integers away from K. This paper is devoted to the computation of the groups ψ_n^K of piecewise-linear Λ-homology n-spheres, modulo H_K-cobordism.

In their classic paper **[12]**, Kervaire and Milnor computed the group of h-cobordism classes of smooth homotopy spheres. This was generalized to homology spheres by Kervaire **[11]**. In two independent contributions, the group of H_K-cobordism classes of smooth Λ-homology spheres was computed by Alexander, Hamrick and Vick **[1]**, in the case $K = \{\text{odd primes}\}$ and by Barge, Lannes, Latour and Vogel **[5]** in the arbitrary case.

In §2, we define ψ_n^K and the corresponding group θ_n^K of Λ-homotopy spheres, requiring Λ-homology spheres and H_K-cobordisms to be simply connected. We show that if $2 \notin K$, then $\theta_4^K = \psi_4^K = 0$ and $\theta_n^K \cong \psi_n^K$ if $n \geq 5$.

In §3, we construct an embedding $L_{n+1}(1; \Lambda)/L_{n+1}(1) \to \theta_n^K$ (or ψ_n^K) and prove

$$\psi_n^K \otimes \Lambda \cong \left(L_{n+1}(1; \Lambda)/L_{n+1}(1)\right) \otimes \Lambda \quad \text{for } n \geq 5.$$

We then show that every Λ-homology sphere is stably K-parallelizable. This is done by computing the mod(K) homotopy groups of the fiber of $B\widetilde{\mathrm{PL}} \to BH(K)$, where $H(K)$ is the structure monoid of Λ-homology cobordism bundles.

In §4, we prove our main result: If $n \geq 4$, $2 \notin K$, then

$$\theta_n^K = \begin{cases} 0, & n \not\equiv 3 \bmod(4), \\ \overline{W}(\Lambda, \mathbf{Z}) \oplus \bigoplus_{\pi(k)-1} \Lambda/\mathbf{Z}, & n = 4k-1, \end{cases}$$

modulo the Serre class of finite K-torsion groups.

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Three dimensional Λ-homology spheres are considered in §5. We construct elements of ψ_3^K corresponding to the mod(K) surgery obstructions as above, and exhibit an epimorphism $\alpha_K: \psi_3^K \to \mathbf{Z}/(16/a_K)\mathbf{Z}$. Finally, we compute the fundamental groups of these manifolds.

2. Groups of homotopy and homology spheres. Let H = DIFF, PL or TOP. A closed H n-manifold Σ is an H *Λ-homology sphere* if $H_*(\Sigma; \Lambda) \cong H_*(S^n; \Lambda)$. Note that Σ is orientable since $H_n(\Sigma) \otimes \Lambda = \Lambda$. An H Λ-homology sphere Σ is an H *Λ-homotopy sphere* if $\pi_1(\Sigma) = 0$.

A cobordism $(W; M, M')$ is an *H_K-cobordism* if $H_*(W, M; \Lambda) = 0$; an H_K-cobordism is an *h_K-cobordism* if, in addition, $\pi_1(M) \cong \pi_1(W) \cong \pi_1(M')$.

Let $\psi_n^K(H)$ be the set of H_K-cobordism classes of oriented H Λ-homology n-spheres, and $\theta_n^K(H)$ the set of h_K-cobordism classes of oriented H Λ-homotopy n-spheres. Both $\psi_n^K(H)$ and $\theta_n^K(H)$ are abelian groups under the operation of connected sum. There is clearly a homomorphism $\theta_n^K(H) \to \psi_n^K(H)$. We let $\theta_n^K = \theta_n^K(\mathrm{PL})$, $\psi_n^K = \psi_n^K(\mathrm{PL})$.

An oriented H-manifold M is *stably K-parallelizable* if the map

$$M \xrightarrow{\nu_M} BSH \to (BSH)_K$$

is null-homotopic. We have the following mild generalization of Lemma 1.1 of [**1**]:

LEMMA 2.1. *Let Σ be a smooth Λ-homology sphere. Then Σ is stably K-parallelizable.*

The proof is immediate from [**1**]. Compare [**12**, Theorem 3.1], and [**11**, Theorem 3].

PROPOSITION 2.2. *Suppose π is a finitely presented group so that $H_1(\pi; \Lambda) = H_2(\pi; \Lambda) = 0$. Then for $n \geqslant 5$, there is a smooth Λ-homology sphere Σ^n with $\pi_1(\Sigma) = \pi$.*

PROOF. Suppose M is a smooth manifold. Then

$$\begin{aligned} H_2(\pi_1(M); \Lambda) &\cong H_2(\pi_1(M)) \otimes \Lambda \quad \text{by the universal coefficient theorem, since } \Lambda \text{ is torsion free} \\ &\cong (H_2(M)/\mathrm{Im}(h)) \otimes \Lambda \quad \text{where } h \text{ is the Hurewicz homomorphism, by [8]} \\ &\cong H_2(M; \Lambda)/\mathrm{Im}(h_K), \quad h_K \text{ the composition } \pi_2(M) \xrightarrow{h} H_2(M) \to H_2(M; \Lambda). \end{aligned}$$

The result now follows exactly as in [**11**, Theorem 1].

Our main result of this section is

THEOREM 2.3. *Suppose* $2 \notin K$. *Then*:
(i) $\theta_n^K \cong \psi_n^K$ *for* $n \geqslant 5$;
(ii) $\theta_4^K = \psi_4^K = 0$.

PROOF. We prove (ii) first. Let Σ^4 be a PL Λ-homology sphere. By [9], Σ is smoothable. Furthermore, Σ has trivial normal bundle, since $H_2(\Sigma; Z/2Z) = H_3(\Sigma; Z/2Z) = 0$, and the obstruction in $H^4(\Sigma; Z)$ is given by a multiple of $p_1(\Sigma) = 3\ \mathrm{Sign}(\Sigma) = 0$. Thus Σ bounds a parallelizable manifold, since $\Omega_4^{\mathrm{fr}} = 0$. Since $L_5(1; \Lambda) = 0$, by [3] $\theta_4^K = 0$. To see that $\psi_4^K = 0$, let $\mathcal{F}: \Lambda[\pi_1(\Sigma)] \to \Lambda$ be the coefficient homomorphism and $\Gamma_5(\mathcal{F})$ the surgery obstruction group of [7]. By [7], $\Gamma_5(\mathcal{F}) = 0$, and Σ bounds a Λ-acyclic manifold.

To prove (i) assume $n \geqslant 5$ and Σ^n is a smooth Λ-homology sphere. By Lemma 2.1, Σ is stably K-parallelizable, and so $T_\Sigma \in KO(\Sigma)$ has order in Λ. Let $\alpha_1, \ldots, \alpha_m$ be generators for $\pi_1(\Sigma)$, represented by disjoint embeddings $\phi_i: S^1 \to \Sigma$. Then $\phi_i^*(T_\Sigma) \in KO(S^1) = Z/2Z$ has odd order, and so the normal bundle of ϕ_i is trivial. Attach handles along $\mathrm{Im}(\phi_i)$; let N be the trace of these surgeries.

Both N and $\Sigma' = \partial_+ N$ and stably K-parallelizable, and

$$\pi_1(\Sigma') = 0, \quad H_i(\Sigma'; \Lambda) = 0, \qquad 3 \leqslant i \leqslant n-3,$$

and $H_2(\Sigma'; \Lambda)$, $H_{n-2}(\Sigma'; \Lambda)$ are free Λ-modules of rank m. Do surgery on a basis for $H_2(\Sigma'; \Lambda)$; the resultant is a Λ-homotopy sphere which is H_K-cobordant to Σ.

Now, if Σ is not smooth, then Σ admits a smoothing in a neighborhood of its 2-skeleton, and the argument goes through as above.

Suppose finally that Σ is a Λ-homotopy sphere and W is an H_K-cobordism to S^n. Then W is stably K-parallelizable and we may do surgery on a set of generators of $\pi_1(W) \cong \pi_1(W, \Sigma)$. Let W' be the resultant. Then the result of doing surgery on a basis for $H_2(W', \Sigma: \Lambda)$ gives an H_K-cobordism between Σ and S^n.

3. Stable K-parallelizability and $\psi_n^K \otimes \Lambda$. Recall that the simply connected surgery obstruction groups with coefficients in Λ, as computed in [3], [4], are given by

$$L_n(1; \Lambda) = \begin{cases} 0, & n \text{ odd}, \\ (\mathbf{Z}/2\mathbf{Z}) \otimes \Lambda, & n \equiv 2 \bmod(4), \\ \overline{W}(\Lambda), & n \equiv 0 \bmod(4), \end{cases}$$

where $\overline{W}(\Lambda) \subseteq \mathbf{Z} \oplus \bigoplus_{p \in K} W(\mathbf{F}_p)$ is the Witt-Wall group of even quadratic forms over Λ, modulo kernels, and

$$W(\mathbf{F}_p) = \begin{cases} \mathbf{Z}/4\mathbf{Z}, & p \equiv 3 \bmod(4), \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, & p \equiv 1 \bmod(4), \\ \mathbf{Z}/2\mathbf{Z}, & p = 2. \end{cases}$$

The embedding above is given by $(1/a_K)$Sign and the Hasse-Minkowski invariants β_p: $W(\mathbf{Q}) \to W(\mathbf{F}_p)$, where

$$a_K = \begin{cases} 1, & K \equiv 0 \bmod(2), \\ 2, & K \equiv 3 \bmod(4), K \not\equiv 0 \bmod(2), \\ 4, & K \neq \varnothing, K \not\equiv 3 \bmod(4), K \not\equiv 0 \bmod(2), \\ 8, & K = \varnothing. \end{cases}$$

(We write $K \equiv a \bmod(b)$ if $p \equiv a \bmod(b)$ for some $p \in K$.)

We have the following result from **[3]**.

PROPOSITION 3.1. *Let $x \in \overline{W}(\Lambda)$ and $k > 1$. Then there is a smooth manifold triad $(M; \partial_+ M, \partial_- M)$ and a normal map ϕ: $(M; \partial_+ M, \partial_- M) \to (S^{4k-1} \times I; S^{4k-1} \times 0, S^{4k-1} \times 1)$ so that*

(i) *$\phi | \partial_+ M$ is the identity,*

(ii) *$\pi_1(\partial_- M) = 0$ and $\phi | \partial_- M$ induces isomorphisms on Λ-homology, and*

(iii) *the cup product pairing of M is equivalent to x in $\overline{W}(\Lambda)$.*

We let $\Sigma_x = \partial_- M$ and $M_x = M \cup c(\partial_+ M) \cup c(\partial_- M)$; Σ_x is a smooth Λ-homotopy $(4k-1)$-sphere and M_x is a closed Λ-homology manifold.

Let $\Omega^{G,K}{}_*$ be the cobordism group of oriented Λ-Poincaré complexes, and $MSG(K)$ the Thom spectrum associated to Λ-spherical fibrations **[19]**. Since closed Λ-homology manifolds satisfy Poincaré duality with Λ-coefficients, the spaces M_x generate a subgroup A_{4k} of $\Omega^{G,K}{}_*$. Let $A_{2n+1} = 0$ and $A_{4k+2} = (\mathbf{Z}/2\mathbf{Z}) \otimes \Lambda$.

PROPOSITION 3.2. *$A_{4k} \cong \overline{W}(\Lambda)$ and for $n \geqslant 5$, there is an exact sequence*

$$0 \to A_n \to \Omega_n^{G,K} \to \pi_n(MSG(K)) \to 0.$$

PROOF. Define f: $\overline{W}(\Lambda) \to A_{4k}$ by $f(x) = [M_x]$. By an elementary surgery argument, f is a homomorphism, which is clearly surjective. Furthermore, if M_x is a boundary, then the standard argument (e.g. **[6**, Theorem III.2.4**]**) shows that x is a kernel, so that f is injective. The second statement is immediate from **[10**, §7.11**]**.

Let $\overline{W}(\Lambda, \mathbf{Z}) = \operatorname{coker}(\overline{W}(\mathbf{Z}) \to \overline{W}(\Lambda))$.

PROPOSITION 3.3. *There is an embedding $\overline{W}(\Lambda, \mathbf{Z}) \to \theta_{4k-1}^K$, $k > 1$.*

PROOF. Define r: $\overline{W}(\Lambda) \to \theta_{4k-1}^K$ by $r(x) = [\Sigma_x]$. Surgery arguments show that r is a well-defined homomorphism. Suppose $r(x) = 0$. Then there is an

h_K-cobordism W from Σ_x to S^{4k-1}. Furthermore, the quadratic form on $H^{2k}(N_x; Q)$ represents x, where $N_x = D^{4k} \cup M \cup W \cup D^{4k}$ and M is the manifold of Proposition 3.1 associated to x. Since N_x is a closed manifold, $x \in \overline{W}(\mathbf{Z})$.

REMARK. The same proof shows that $\overline{W}(\Lambda, \mathbf{Z})$ embeds in ψ_{4k-1}^K, $k > 1$.

PROPOSITION 3.4. *Let Σ^n be a stably parallelizable Λ-homotopy sphere, $n \geq 4$. Then* $[\Sigma] = 0$ *in* θ_n^K *if* $n \not\equiv 3$ mod(4), *and is h_K-cobordant to some Σ_x otherwise.*

PROOF. Let $f\colon \Sigma \to S^n$ collapse the exterior of a disc to a point. Then f and 1_{S^n} determine the same PL-normal invariant, and so are normally cobordant, by, say, $F\colon W \to S^n \times I$. By adding a certain manifold M, as constructed above, to W, we may assume the surgery obstruction of F vanishes, so that Σ is h_K-cobordant to ∂M. Clearly $\partial M = S^{n-1}$ if $n \not\equiv 3$ mod(4) and some Σ_x otherwise.

The following is implicit in Quinn [**16**].

THEOREM 3.5. *For $n \geq 5$, $\psi_n^K \otimes \Lambda \cong (L_{n+1}(1; \Lambda)/L_{n+1}(1)) \otimes \Lambda$.*

PROOF. We sketch the proof. Let $\mathcal{S}_{\mathrm{PL}}^K(S^n)$ be the set of Λ-homology triangulations of S^n, i.e., degree 1 Λ-homology equivalences $f\colon M^n \to S^n$ modulo H_K-cobordisms. Define $\phi\colon \psi_n^K \to \mathcal{S}_{\mathrm{PL}}^K(S^n)$ by sending Σ to the map $\Sigma \to S^n$, collapsing the exterior of an embedded n-disc to a point. (If W is an H_K-cobordism between Σ and Σ', then collapsing the exterior of a regular neighborhood of an embedded path between Σ and Σ' shows that ϕ is well defined.) ϕ is a clearly a bijection.

Let $\mathfrak{N}_{\mathrm{PL}}^K(S^n)$ be the set of K-normal invariants of S^n, cobordism classes of degree 1 mappings $f\colon M^n \to S^n$ together with a factorization

$$\begin{array}{ccc} M & \xrightarrow{\nu_M} & (B\mathrm{SPL})_K \\ {\scriptstyle f}\downarrow & \nearrow & \\ S^n & & \end{array}$$

By [**16**], there are exact sequences for $n \geq 5$,

$$\cdots \to L_{n+1}(1; \Lambda) \to \mathcal{S}_{\mathrm{PL}}^K(S^n) \to \mathfrak{N}_{\mathrm{PL}}^K(S^n) \to L_n(1; \Lambda),$$

$$\cdots \to \Omega_n^K \to \mathfrak{N}_{\mathrm{PL}}^K(S^n) \to \pi_n(G/\mathrm{PL}) \otimes \Lambda \to \Omega_{n-1}^K \to \cdots,$$

where $\Omega_n^K = \pi_n^S(MS\mathrm{PL}^{(K)})$, and $MS\mathrm{PL}^{(K)}$ is the fiber of $MS\mathrm{PL} \to (MS\mathrm{PL})_K$. Since $\Omega^K_* \otimes \Lambda = 0$, and $\pi_n(G/\mathrm{PL}) = L_n(1)$ for $n \geq 5$, the result follows.

To prove that PL Λ-homology spheres are stably K-parallelizable, we introduce Λ-homology cobordism bundles.

A polyhedron M is called a Λ-*homology manifold* of dimension n if M has a subdivision M' so that $\tilde{H}_*(Lk(x, M'); \Lambda) \cong \tilde{H}_*(S^{n-1}; \Lambda)$ or 0. The *boundary* of M,

$$\partial M = \left\{x \in M' : \tilde{H}_*(Lk(x, M'); \Lambda) = 0\right\}$$

is a Λ-homology manifold of dimension $n - 1$.

A Λ-*homology n-sphere* is a Λ-homology n-manifold Σ so that $H_*(\Sigma; \Lambda) \cong H_*(S^n; \Lambda)$; a Λ-*homology n-disc* is a compact Λ-acyclic Λ-homology n-manifold Δ. The prefix "PL" indicates that Σ or Δ is a PL manifold. A Λ *n-cell* is the cone cM over a Λ-homology $(n-1)$-sphere or $(n-1)$-disc M; such a Λ n-cell is a Λ-homology n-manifold with boundary M or $M \cup c(\partial M)$. An H_K-*cobordism* is a Λ-homology manifold triad $(W; M_+, M_-)$ with $H_*(W; M_\pm; \Lambda) = 0$. (Again the prefix "PL" means that W is a PL-manifold.)

A Λ-*cell decomposition* of a simplicial complex X is a collection $\mathcal{O}$ of subpolyhedra of X so that

(i) each $\Delta \in \mathcal{O}$ is a Λ-cell,

(ii) X has a subdivision X' so that every simplex of X' lies in the interior of a unique element of $\mathcal{O}$, and

(iii) if $\Delta \in \mathcal{O}$, $\partial\Delta$ is a union of elements of $\mathcal{O}$.

Let X be a simplicial complex with a Λ-cell decomposition $\mathcal{O}$. A Λ-*homology cobordism* (*n-sphere*) *bundle* ξ over X is a complex $E = E(\xi)$ over $\mathcal{O}$ (see [15, p. 96]) so that for each $\Delta^m \in \mathcal{O}$:

(i) $E(\Delta)$ is a Λ-homology $(n + m)$-manifold with

$$\partial E(\Delta) = E(\partial\Delta) = \bigcup_{\substack{\Delta_0 \in \mathcal{O}|\Delta \\ \Delta_0 \neq \Delta}} E(\Delta_0),$$

and

(ii) there is a complex W over $\mathcal{O}|\Delta$ so that $W(\Delta_0)$ is an H_K-cobordism between $E(\Delta_0)$ and $\Delta_0 \times S^n$ for each $\Delta_0 \in \mathcal{O}|\Delta$.

Here $\mathcal{O}|\Delta$ denotes the Λ-cell decomposition of Δ consisting of those $\Delta_0 \in \mathcal{O}$ with $\Delta_0 \subset \Delta$.

Two Λ-homology cobordism bundles ξ^n, η^n over X are *isomorphic*, written, $\xi \cong \eta$, if there is a complex G over $\mathcal{O}$ so that for each $\Delta \in \mathcal{O}$, $G(\Delta)$ is an H_K-cobordism between $E(\xi)(\Delta)$ and $E(\eta)(\Delta)$.

These bundles satisfy the analogous properties that the homology cobordism bundles of [15] enjoys. In particular, isomorphism classes on Λ-homology cobordism n-sphere bundles are classified by $BH(K)_{n+1}$, where $H(K)_{n+1}$ is the Δ-monoid with i-simplexes given by isomorphisms of the trivial bundle $\Delta^i \times S^n$ over Δ^i. There are stabilization maps $BH(K)_n \to BH(K)_{n+1}$ and we let $BH(K) = \lim BH(K)_n$.

Let $\overline{\mathrm{PL}}_n$ be the Δ-set with i-simplexes PL n-block bundles over $\Delta^i \times I$ which are trivial over $\Delta^i \times \dot{I}$. Then $\overline{\mathrm{PL}}_n$ and $\widetilde{\mathrm{PL}}_n$ are homotopy equivalent (see [14]), and $\overline{\mathrm{PL}}_n \subset H(K)_n$. Therefore there is a natural map $B\widetilde{\mathrm{PL}} \to BH(K)$, with fiber denoted by $H(K)/\widetilde{\mathrm{PL}}$.

THEOREM 3.6. $\pi_n(H(K)/\widetilde{\mathrm{PL}}) \otimes \Lambda \cong \psi_n^K \otimes \Lambda$.

The proof is immediate from the following two lemmas. Recall from [18] that a Λ-acyclic resolution $f: N^n \to M^n$ between Λ-homology manifolds is a proper, PL surjection so that $\tilde{H}_*(f^{-1}(x); \Lambda) = 0$ for every $x \in M$, and $f|\partial N: \partial N \to \partial M$ is also a Λ-acyclic resolution, and that if M is orientable and ∂M is a PL-manifold, then there is a Λ-acyclic resolution, rel(∂M), to a PL-manifold N provided obstructions $\mu_j \in H_j(M; \psi_{n-j-1}^K)$, $j = n - 1, \ldots, 0$, vanish.

Let $\mathrm{PL}H(K)_n$ be the Δ-set of which a typical i-simplex is a block-preserving PL H_K-cobordism between $\Delta^i \times S^{n-1}$ and itself. We have $\overline{\mathrm{PL}}_n \subset \mathrm{PL}H(K)_n \subset H(K)_n$.

LEMMA 3.7. $\pi_n(H(K)/\mathrm{PL}H(K)) \otimes \Lambda \cong \psi_n^K \otimes \Lambda$.

PROOF. Let $x \in \pi_n(H(K)_i/\mathrm{PL}H(K)_i)$. Then x is represented by a Λ-homology cobordism i-sphere bundle W over $\Delta^n \times I$, trivial over $\Delta^n \times \dot{I}$, and a PL H_K-cobordism over $\Delta^{n-1} \times I$; W is a Λ-homology $(n + i + 1)$-manifold with PL boundary, and $H_*(W; \Lambda) \cong H_*(S^i; \Lambda)$.

Let $\mu_j(x) \in H_j(W; \psi_{n+i-j}^K)$ be the first nonzero obstruction to resolving W rel(∂W). If $j > i$, then $k\mu_j(x) = 0$ for some $k \in \Lambda^{\cdot}$, and by naturality of the obstructions, $\mu_j(kx) = 0$. Continuing in this way, there are two obstructions $\mu_i \in \psi_n^K \otimes \Lambda$, $\mu_0 \in \psi_{n+i}^K \otimes \Lambda$ to resolving an isomorphism representing kx for some $k \in \Lambda^{\cdot}$.

Define ϕ_i: $\pi_n(H(K)_i/\mathrm{PL}H(K)_i) \otimes \Lambda \to \psi_n^K \otimes \Lambda$ by $\phi_i(x) = k^{-1}\mu_i(kx)$, where k is chosen as above. It follows easily that ϕ_i is a well-defined homomorphism and that

$$\begin{array}{ccc} \pi_n(H(K)_i/\mathrm{PL}H(K)_i) \otimes \Lambda & \xrightarrow{\phi_i} & \psi_n^K \otimes \Lambda \\ \downarrow & & \downarrow = \\ \pi_n(H(K)_{i+1}/\mathrm{PL}H(K)_{i+1}) \otimes \Lambda & \xrightarrow{\phi_{i+1}} & \psi_n^K \otimes \Lambda \end{array}$$

commutes. Define

$$\phi = \lim \phi_i: \pi_n(H(K)/\mathrm{PL}H(K)) \otimes \Lambda \to \psi_n^K \otimes \Lambda.$$

To see that ϕ is injective, let $x \in \ker(\phi)$, and represent kx by a Λ-homology cobordism i-sphere bundle with $n + i \not\equiv 3$ mod(4) and $\mu_j(kx) = 0$ for $j > i$. Then $\phi(x) = k^{-1}\mu_j(kx) = 0$, so that $\mu_i(kx)$ is K-torsion. Since $n + i \not\equiv 3$ mod(4), $\psi_{n+i}^K \otimes \Lambda = 0$ by Theorem 3.5, and so all obstructions $\mu_i, \mu_{i-1}, \ldots, \mu_0$ are K-torsion. As above, there exists $k' \in \Lambda$ so that $k'kx$ is represented by a bundle W with all obstructions to resolving W rel(∂W) vanishing. Thus there is a PL-manifold V and a Λ-acyclic resolution f:

$V \to W$, identity on the boundary. But V represents 0 in $\pi_n(H(K)_i/\mathrm{PL}H(K)_i)$, and the mapping cylinder of f represents a homotopy from $k'kx$ to V. Therefore $x = 0$.

Finally, ϕ is surjective: Let $y \in \psi_n^K \otimes \Lambda$ and Σ^n a PL Λ-homology sphere representing ky for some $k \in \dot{\Lambda}$. By choosing an embedded n-disc in Σ, we can regard $c\Sigma$ as a space over $\Delta^n \times I$, as in **[14]**, and $c\Sigma \times S^i$ represents an element $x \in \pi_n(H(K)_i/\mathrm{PL}H(K)_i)$ with a single resolution obstruction $\mu_j(x) = [\Sigma]$. Then $\phi(x/k) = y$.

LEMMA 3.8. $\pi_n(\mathrm{PL}H(K)/\widetilde{\mathrm{PL}}) \otimes \Lambda = 0$.

PROOF. Let $x \in \pi_n(\mathrm{PL}H(K)/\widetilde{\mathrm{PL}})$ be represented by a PL H_K-cobordism W of $\Delta^n \times S^i$ that is a PL block bundle over $\dot{\Delta}^n \times I$ and the product bundle over $\Delta^{n-1} \times I$, with $n + i \not\equiv 2 \bmod(4)$. Extend $W|\partial(\Delta^n \times I)$ to a disc bundle V, and let $\Sigma = W \cup V$. Then Σ is a PL Λ-homology $(n + i + 1)$-sphere, and by Theorem 3.5, $\#_k\Sigma$ bounds a Λ-acyclic manifold H for some $k \in \dot{\Lambda}$. By replacing x by kx, we may assume $W \cup V$ bounds a Λ-acyclic manifold H; by Corollary 3.3 of **[5]** we may further assume that $\pi_1(H) = 0$.

Let i: $\partial(\Delta^n \times I) \to V$ be the zero section. By the Hurewicz theorem, $\pi_n(H) \otimes \Lambda = 0$, and so $k'[i] = 0$ in $\pi_n(H)$ for some $k' \in \Lambda$. Again, replacing x with $k'x$, we may assume i is null-homotopic. The remainder of the proof now follows exactly as the proof of Lemma 2.1 of **[14]**.

THEOREM 3.9. *Let Σ^n be a* PL *Λ-homology sphere. Then Σ is stably K-parallelizable.*

PROOF. *Case* 1. $n \not\equiv 3 \bmod(4)$.

Let $\Sigma \xrightarrow{\nu} BSH(K)$ classify the stable normal H_K-homology cobordism bundle of Σ. Since Σ bounds the contractible Λ-homology manifold $c\Sigma$, ν is null-homotopic. The obstructions to lifting this null-homotopy to $(BS\mathrm{PL})_K$ lie in

$$H^i\left(\Sigma; \pi_i(SH(K)/S\widetilde{\mathrm{PL}}) \otimes \Lambda\right) = \begin{cases} 0, & i \neq n, \\ \psi_n^K \otimes \Lambda, & i = n, \end{cases}$$

by Theorem 3.6. Since $n \not\equiv 3 \bmod(4)$, $\psi_n^K \otimes \Lambda = 0$.

Case 2. $n \equiv 3 \bmod(4)$.

The argument of Case 1 shows that $\Sigma \xrightarrow{\nu} (BSG)_K$ is null-homotopic, and this null-homotopy lifts to $(BS\mathrm{PL})_K$ since $\pi_n(G/\mathrm{PL})_K = 0$.

4. The calculation of θ_n^K, $2 \notin K$. In this section, we sharpen Theorem 3.4 in the odd case.

THEOREM 4.1. *Let $2 \notin K$. Then for $n \geqslant 4$,*

$$\theta_n^K = \begin{cases} 0, & n \not\equiv 3 \bmod(4), \\ \overline{W}(\Lambda, \mathbf{Z}) \otimes \bigoplus_{\pi(k)-1} \Lambda/\mathbf{Z}, & n = 4k-1, \end{cases}$$

modulo the Serre class of finite K-torsion groups.

Here the $\pi(k)$ denotes the number of partitions of k.

PROOF. Let $\mathbf{T}^H$ denote the spectrum for the cobordism group $\Omega^{\mathrm{fr},K}{}_*(H)$ of stably K-parallelizable H-manifolds, $H = \mathrm{DIFF}$ or PL. Let **MSH** be the Thom spectrum associated to BSH_n and **S** the sphere spectrum.

There is a homotopy equivalence $\mathbf{T}^H/\mathbf{S} \to (\mathbf{MSH}/\mathbf{S})^{(K)}$ (compare [**2**, p. 85]), and we have an exact sequence

$$\begin{array}{ccccccc} 0 \to & \Omega_n^{SO} \to & \Omega_n^{SO,\mathrm{fr}} \to & \Omega_{n-1}^{\mathrm{fr}}(\mathrm{DIFF}) \to 0 \\ & \alpha\downarrow & \beta\downarrow & \gamma\downarrow \\ 0 \to & \Omega_n^{SPL} \to & \Omega_n^{SPL,\mathrm{fr}} \to & \Omega_{n-1}^{\mathrm{fr}}(\mathrm{PL}) \to 0 \end{array}$$

where $\Omega_n^{SH,\mathrm{fr}} = \pi_n(\mathbf{MSH}/\mathbf{S})$ is the cobordism group of oriented H-manifolds with framed boundaries.

By [**9**], γ is an isomorphism. Also, α is injective with cokernel a finite group.

Let **F** be the fiber of $\mathbf{MSO}/\mathbf{S} \to \mathbf{MSPL}/\mathbf{S}$; then $\mathbf{F}^{(K)}$ is the fiber of $\mathbf{T}^0/\mathbf{S} \to \mathbf{T}^{\mathrm{PL}}/\mathbf{S}$. Furthermore, the map $\pi_n(\mathbf{F}^{(K)}) \to \pi_n(\mathbf{T}^0/\mathbf{S})$ is 0 for all n. This is clear if $n \not\equiv 3$ mod(4), since $\pi_n(\mathbf{T}^0/\mathbf{S}) \to \pi_n(\mathbf{MSO}/\mathbf{S})$ is then injective and $\pi_n(\mathbf{F}) \to \pi_n(\mathbf{MSO}/\mathbf{S})$ is 0. If $n \equiv 3$ mod(4), then we have

$$\begin{aligned} K\text{-torsion in } \Omega_{n+1}^{SPL,\mathrm{fr}}/\Omega_{n+1}^{SO,\mathrm{fr}} &= \pi_n(\mathbf{F}^{(K)}) \\ &\to \pi_n(\mathbf{MSO}/\mathbf{S}) \\ &= \mathrm{coker}\left[\Omega_{n+1}^{SO,\mathrm{fr}} \to \left(\Omega_{n+1}^{SO,\mathrm{fr}}\right)_K\right] \end{aligned}$$

which is clearly 0.

Thus we have an exact sequence

$$0 \to \pi_n(\mathbf{T}^0/\mathbf{S}) \to \pi_n(\mathbf{T}^{\mathrm{PL}}/\mathbf{S}) \to K\text{-torsion in } \Omega_n^{SPL,\mathrm{fr}}/\Omega_n^{SO,\mathrm{fr}} \to 0.$$

The homotopy sequences of the pairs $(\mathbf{T}^0, \mathbf{S})$, $(\mathbf{T}^{\mathrm{PL}}, \mathbf{S})$ now imply that there is an exact sequence

$$0 \to \Omega_n^{\mathrm{fr},K}(\mathrm{DIFF}) \to \Omega_n^{\mathrm{fr},K}(\mathrm{PL}) \to K\text{-torsion in } \Omega_n^{SPL,\mathrm{fr}}/\Omega_n^{SO,\mathrm{fr}} \to 0.$$

Consider the commutative diagram

$$\begin{array}{cccccc} 0 \to & bP_{n+1}^K & \longrightarrow & \theta_n^K(\mathrm{DIFF}) \to & \Omega_n^{\mathrm{fr},K}(\mathrm{DIFF}) \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & L_{n+1}(1;\Lambda)/L_{n+1}(1) & \to & \theta_n^K \longrightarrow & \Omega_n^{\mathrm{fr},K}(\mathrm{PL}) \end{array}$$

where bP_{n+1}^K is the subgroup of smooth Λ-homology spheres that bound K-parallelizable manifolds. The top row is clearly exact, and the map bP_{n+1}^K

$\to L_{n+1}(1;\Lambda)/L_{n+1}(1)$, defined by the surgery obstruction, is surjective, since the manifolds Σ_x are smooth, with kernel a finite group [1]. Again by [16],

$$L_{n+1}(1;\Lambda) \to \psi_n^K \to \mathfrak{N}_{\mathrm{PL}}^K(S^n)$$

is exact, and so

$$\ker(\theta_n^K \to \Omega_n^{\mathrm{fr},K}(\mathrm{PL})) = \ker(\psi_n^K \to \mathfrak{N}_{\mathrm{PL}}^K(S^n)) = L_{n+1}(1;\Lambda)/L_{n+1}(1).$$

Thus $\theta_n^K(\mathrm{DIFF}) \to \theta_n^K$ has cokernel a finite K-torsion group. Furthermore, by [1], there is an exact sequence

$$\theta_n(\mathrm{DIFF}) \to \theta_n^K(\mathrm{DIFF}) \to L_{n+1}(1;\Lambda)/L_{n+1}(1) \oplus G_n \to 0$$

where

$$G_n = \begin{cases} 0, & n \not\equiv 3 \bmod(4), \\ \bigoplus_{\pi(k)-1} \Lambda/\mathbf{Z}, & n = 4k-1. \end{cases}$$

Since $\theta_n(\mathrm{DIFF}) \to \theta_n^K$ is 0, θ_n^K is given as stated.

5. The 3-dimensional case. In general, the methods of the preceding sections do not apply in dimension 3, due to the lack of a well-behaved surgery theory. In this section we investigate the Λ-homology 3-spheres obtained from the plumbing theorem and compute their fundamental groups.

We define the *α_K-invariant*, $2 \notin K$, as follows. Let Σ^3 be a PL Λ-homology sphere. By Lemma 2.1, there is a homomorphism $\psi_3^K \to \Omega_3^{\mathrm{fr},K} \cong \Lambda/\mathbf{Z}$ [2]. Thus for some integer $k \in \Lambda^{\cdot}$, $\#_k\Sigma$ bounds a K-parallelizable manifold W. Define $k\alpha_K(\Sigma) = (1/a_K)\mathrm{Sign}(W) \bmod(16/a_K)$. We show α_K is well defined. Suppose $\#_{k_i}\Sigma = \partial W_i$, $i = 1, 2$. Let $W = (\#_{k_1}W_2) \cup (\#_{k_2} - W_1)$ identified along the common boundary $\#_{k_1+k_2}\Sigma$. Then

$$k_1\,\mathrm{Sign}(W_2) - k_2\,\mathrm{Sign}(W_1) = \mathrm{Sign}(W) = 0 \bmod(16) \quad \text{by } [\mathbf{17}].$$

Let A be a symmetric integral matrix with even diagonal entries and $\det(A) \in \Lambda^{\cdot}$. We may apply the plumbing theorem to construct a smooth manifold M_A^4 with intersection pairing A and $\Sigma_A^3 = \partial M_A^4$ a Λ-homology sphere (this is done explicitly in [11] for $K = \varnothing$). By definition of a_K, there is a matrix A as above with $\mathrm{Sign}(A) = a_K$. Thus we have

PROPOSITION 5.1. *There is a surjection $\alpha_K\colon \psi_3^K \to \mathbf{Z}/(16/a_K)\mathbf{Z}$.*

We now compute $\pi_1(\Sigma_A)$ for the set of generators A of $\overline{W}(\Lambda)$ described in [4].

(1) A = the Milnor matrix: By [11],

$$\pi_1(\Sigma_A) = \langle x, y\colon x^3 = y^2 = (xy)^5\rangle.$$

(2) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2k\end{pmatrix}$: Let S_1, S_2 be the attaching spheres for the 2-handles of M_A, and x, y the loops indicated below:

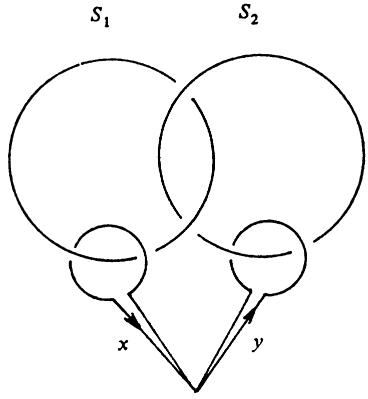

FIGURE 1

We clearly have that $\pi_1(\Sigma_A)$ is generated by x and y, and $x^{2k} = y = xy^{-1}$. Thus $\pi_1(\Sigma_A) \cong \mathbf{Z}/(4k-1)\mathbf{Z}$.

(3) $A = \begin{pmatrix} -2 & 1 \\ 1 & 2k \end{pmatrix}$: As in (2), $\pi_1(\Sigma_A) \cong \mathbf{Z}/(4k+1)\mathbf{Z}$.

(4) $A = \begin{pmatrix} 2a & a \\ a & 2ak \end{pmatrix}$: Letting x and y be as in (2), we have

$$\pi_1(\Sigma_A) \cong \langle x, y \colon x^{2ak} = y^a, y^{2a} = x^a \rangle \cong \mathbf{Z}/(4k-1)a\mathbf{Z}.$$

(5)

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2k \end{bmatrix}:$$

Let S_1, S_2, S_3, S_4 be the attaching spheres and x, y as indicated:

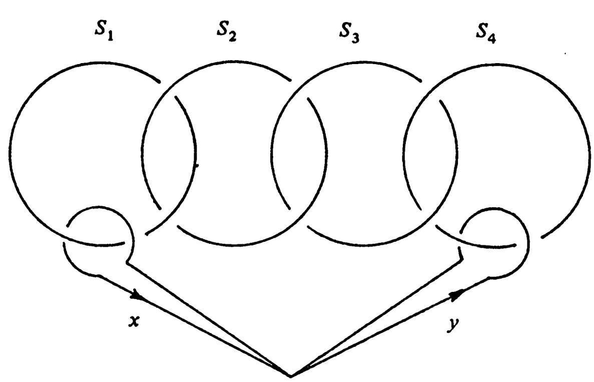

FIGURE 2

Then $\pi_1(\Sigma_A) \cong \langle x, y \colon x^{4k} = y, y^2 = x^3 \rangle \cong \mathbf{Z}/(8k-3)\mathbf{Z}$.

EXAMPLE. Let $p = 4k - 1$ be a prime and M^3 the rational homology sphere obtained by intersecting the unit 5-sphere in $\mathbf{C}^3$ with algebraic variety defined by $z_1^2 + z_2^2 + z_3^p = 0$. The origin is an isolated singularity of this variety and its minimal resolution has the $(p-1) \times (p-1)$ intersection matrix

$$A = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & \bigcirc & \\ & 1 & & & & \\ & & & \ddots & & \\ & \bigcirc & & & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}$$

which can be diagonalized as $\langle -2, -3/2, -4/3, \ldots, -p/(p-1)\rangle$. Thus $\operatorname{Sign}(A) = -p + 1$, $\beta_p(A) = \langle -(p-1)\rangle = 1$, $\beta_{p'}(A) = 0, p' \neq p$. It follows that $M \cong (\#_n \Sigma_1) \# \Sigma_2$, where Σ_1 is the manifold of (1) above, Σ_2 the manifold of (2) (with orientations reversed) and $n = [k/2]$.

REFERENCES

1. J. P. Alexander, G. C. Hamrick and J. W. Vick, *Involutions on homotopy spheres*, Invent. Math. **24** (1974), 35–50.

2. ______, *Cobordism of manifolds with odd order normal bundles*, Invent. Math. **24** (1974), 83–94.

3. G. A. Anderson, *Surgery with coefficients*, Lecture Notes in Math., vol. 591, Springer-Verlag, Berlin and New York, 1977.

4. ______, *Calculation of the surgery obstruction groups* $L_{4k}(1; \mathbf{Z}_P)$ (to appear).

5. J. Barge, J. Lannes, F. Latour and P. Vogel, *Λ-spheres*, Ann. Sci. École Norm. Sup. **4** (1974), 463–506.

6. W. Browder, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65, Springer-Verlag, Berlin, 1972.

7. S. E. Cappel and J. L. Shaneson, *The codimension two placement problem and homology equivalent manifolds*, Ann. of Math. (2) **99** (1974), 277–348.

8. H. Hopf, *Fundmental gruppe und zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1941), 257–309.

9. M. W. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*, Ann. of Math. Studies, no. 80, Princeton Univ. Press, Princeton, N. J., 1975.

10. L. Jones, *Patch spaces: a geometric representation for Poincaré complexes*, Ann. of Math. (2) **98** (1973), 306–343.

11. M. A. Kervaire, *Smooth homology manifolds and their fundamental groups*, Trans. Amer. Math. Soc. **144** (1969), 67–72.

12. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

13. R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, triangulations and smoothings*, Ann. of Math. Studies, no. 88, Princeton Univ. Press, Princeton, N. J., 1977.

14. N. Martin, *On the difference between homology and piecewise-linear bundles*, J. London Math. Soc. **6** (1973), 197–204.

15. N. Martin and C. R. F. Maunder, *Homology cobordism bundles*, Topology **10** (1971), 93–110.

16. F. Quinn, *Semifree group actions and surgery on PL homology manifolds*, Geometric Topology, Lecture Notes in Math., vol. 438, Springer-Verlag, Berlin, 1975.

17. V. A. Rohlin, *A new result in the theory of 4-dimensional manifolds*, Soviet Math. Dokl. **8** (1952), 221–224.

18. D. Sullivan, *Singularities in space*, Liverpool Singularities Symposium. I, Lecture Notes in Math., vol. 208, Springer-Verlag, Berlin, 1971.

19. ______, *Geometric topology: Localization, periodicity and Galois symmetry*, Lecture Notes, M.I.T., 1970.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802